Collocating Local Volatility Model

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QuantLib User Meeting 2016
Düsseldorf
08.12.2016

QuantLib Integration I: Multi-Threading

- The pricing library is mainly JVM based with QuantLib being integrated via SWIG
- Hosted in WebContainers on multiple NUMA machines

  ⇒ Thread-safe code is crucial
- Communication: JSON or XML via http or WebSockets
- Do not share QuantLib objects between threads

  ⇒ use Riccardo’s thread local singleton pattern
- JVM has multi-threaded garbage collector

  ⇒ set QL_ENABLE_THREAD_SAFE_OBSERVER_PATTERN

QuantLib Integration II: Memory Performance

- Important use case: Large number of parallel pricing via PDE
- Observation on large NUMA Windows server:

  Kernel consumes almost all CPU time
- Many new/delete calls with same array size

  ⇒ Malloc does not scale
- Solution: thread local "cache" for Array and Matrix

  ⇒ released memory is kept in thread local list
  ⇒ next Array related malloc reuses thread local memory

QuantLib Integration III: Garbage Collection

Memory Bombshell

```
for (i <- 0 until 1000000)
  new org.quantlib.Array(125000)
```

QuantLib Integration III: Garbage Collection

- Code sniplet is likely to shutdown your JVM
- Memory allocated by QuantLib Array is unknown to JVM
- JVM will not trigger garbage collector even if the OS is already running out of memory
- Workaround from SWIG manual does not work on recent JVMs

Solution from the ’90s

```
for (i <- 0 until 10000000)
  (new org.quantlib.Array(125000)) delete
```

Introduction: Different View on Calibration

Risk neutral pricing with implied probability density function $f(S_t, t)$

$$V_{call}(t, K) = e^{-rT} \int_K^{\infty} (S_t - K) f(S_t, t) dS_t$$

By differentiation of the equation above, the market implied cumulative distribution function $F_{S_t}(s)$ is given by

$$\begin{aligned} F_{S_t}(s) &= Prop(S_t \leq s) = \int_0^s f(S_t, t) dS_t \\ &= 1 + e^{rt} \frac{\partial V_{call}(t, K)}{\partial K} \mid_{K=s} = \; e^{rt} \frac{\partial V_{put}(t, K)}{\partial K} \mid_{K=s} \end{aligned}$$

Model calibration: Tune model parameters to match $F_{S_t}(s)$.

Exact Calibration: Local Volatlility Model

In the Local Volatility (LV) model the volatility $\sigma_{LV}(S, t)$ is a function of spot level S_t and time t. The dynamics of the spot price is given by:

$$dS_t = (r_t - q_t)\, S_t dt + \sigma_{LV}(S_t, t) S_t dW_t$$

$$\sigma_{LV}^2(S, t) = \left. \frac{\frac{\partial C}{\partial T} + (r_t - q_t)\, K \frac{\partial C}{\partial K} + q_t C}{\frac{K^2}{2} \frac{\partial^2 C}{\partial K^2}} \right|_{K=S, T=t}$$

The model is often criticized for its unrealistic volatility dynamics.

The Dupire formula is mathematically appealing but also unstable.

Lech Grzelak: Collocating Local VolatlilityModel

The dynamics of the spot process is given by some kernel process X_t and a deterministic mapping function $g(X, t)$ such that

$$\begin{aligned} S_t &= g(t, X_t) \\ dX_t &= \mu(X_t, t)dt + \sigma(X_t, t)dW_t \end{aligned}$$

The mapping function $g(t, x)$ is chosen such that the cumulative distribution function (CDF) of S_t matches the market implied CDF.

The LV model is a special case of the CLV model with

$$\begin{aligned} g(t, X_t) &\equiv X_t \\ \mu(X_t, t) &= (r_t - q_t)\, X_t \\ \sigma(X_t, t) &= \sigma_{LV}(X_t, t)X_t \end{aligned}$$

Collocating Local Volatlility (CLV) Model

- The choice of the stochastic kernel process X_t does not influence the model prices of vanilla European options as they are given by the market implied terminal CDF.
- The kernel process X_t influences the dynamics of the forward volatility skew generated by the model and therefore the prices of exotic options.
- It is also preferable to choose an analytical trackable process X_t to reduce the computational efforts.

The Mapping Function $g(t, x)$

- Define a set of calibration maturities $T_i, i = 1, \ldots, m$
- Define the set kernel collocation points $x_{i,j} = x_j(T_i), j = 1, \ldots, n$ having n interpolation points per maturity T_i.
- Define the spot colloation points by $s_{i,j} = g(t_i, x_{i,j})$
- Calibration: The CDF of X_t should match the market implied CDF of S_t for all collocation points.

$$
\begin{aligned}
F_{X_{T_i}}\left(x_{i,j}\right) &= F_{S_{T_i}}\left(s_{i,j}\right) = F_{S_{T_i}}\left(g(T_i, x_{i,j})\right) \\
\Rightarrow g(T_i, x_{i,j}) &= F^{-1}_{S_{T_i}}\left(F_{X_{T_i}}(x_{i,j})\right)
\end{aligned}
$$

- $F_{X_{T_i}}$ is analytically known or easy to compute.
- $F^{-1}_{S_{T_i}}(x)$ can be calculated from $F_{S_{T_i}}(s)$ via Brent solver.

The Mapping Function $g(t, x)$

The values of the two dimensional mapping function $g(t, x)$ are known for T_i and collocation points $x_{i,j}$ with $s_{i,j} = g(t_i, x_{i,j})$.

Interpolation scheme:

- Linear interpolation in t:

$$s_j(t) = s_{i,j} + \left(s_{i+1,j} - s_{i,j}\right) \frac{t - T_i}{T_{i+1} - T_i}, \forall t \in [T_i, T_{i+1}]$$

- Lagrange interpolation in x:

$$g(t, x(t)) = \sum_{j=1}^{N} s_j(t) l_j(x(t)),\ l_j(x(t)) = \prod_{k=1, j\neq k}^{N} \frac{x(t) - x_j(t)}{x_k(t) - x_j(t)}$$

The Collocation Points

Aim: Stable interpolation under the probability distribution $X(t)$

$$\mathbb{E}\left[S(T_i)\right] = \mathbb{E}\left[g(T_i, X(T_i))\right] + \epsilon$$

Optimal collocation points are the Gauss quadrature abscissæ of the underlying probability distribution of the kernel process $X(t)$.

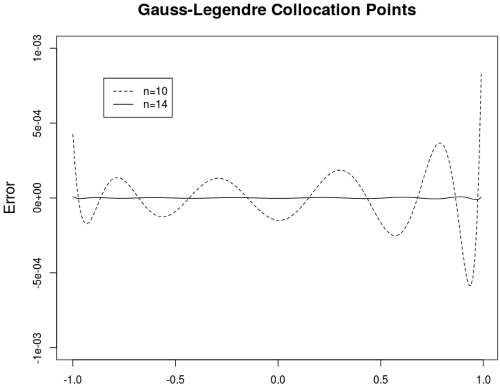
Use the barycentric formulation of the Lagrange interpolation.

The Collocation Points

Example:

- Lagrange interpolation of $f(x) = \frac{e^x}{\cos(x)}$ in $x \in \{-1, 1\}$
- Uniform probability distribution $\Rightarrow$ Gauss-Legendre abscissæ

```
Array x = GaussLegendreIntegration(n).x();
```



The Normal-CLV Model

Choose Ornstein-Uhlenbeck Kernel Process

$$X_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

The exact solution is a normal distribution with

$$\mathbb{E}\left[X(t)\right] = X(t_0)e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right), \mathbb{V}ar\left[X(t)\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

Collocation points are therefore given by

$$x_j(t) = \mathbb{E}\left[X_t\right] + \sqrt{\mathbb{V}ar\left[X_t\right]}x_j^{\mathcal{N}(0,1)}$$

with $x_j^{\mathcal{N}(0,1)} = \mathbf{x}$ given by the Gauss-Hermite abscissæ

```
Array x=std::sqrt(2)*GaussHermiteIntegration(n).x();
```

Pricing under the Normal-CLV Model

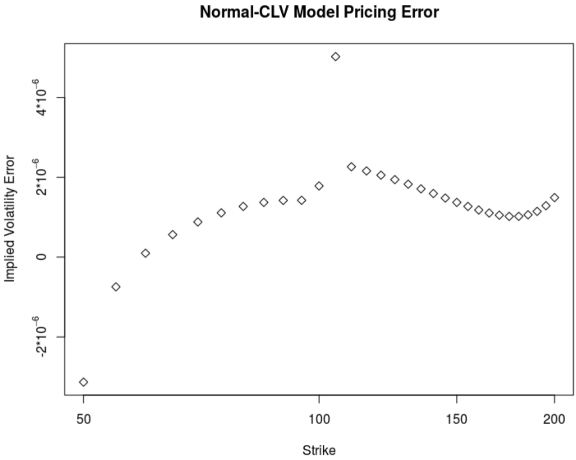
Once the mapping function $g(t, x)$ has been calibrated to the market implied probability distribution pricing is straight forward.

Partial Differential Equation (Feynman-Kac):

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \kappa\left(\theta - x\right)\frac{\partial V}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - rV &= 0 \\
V(T, S) &= \text{Payoff}\left(g(T, X(T)\right)
\end{aligned}
$$

Normal-CLV: Pricing Error for Vanilla Options

- Market prices model: $S_0 = 100, r = 0.1, q = 0.04, \sigma = 0.25, T = 1$
- Normal-CLV process: $\kappa = 1.0, \theta = 0.1, \sigma = 0.5, x_0 = 0.1, n = 10$



Normal-CLV: Forward Volatility Skew

Model Setup:

- Market prices are given by Heston model with

$$S_0 = 100, r = 0.1, q = 0.05,$$
$$\nu_o = 0.09, \kappa = 1.0, \theta = 0.06, \sigma = 0.4, \rho = -0.75$$

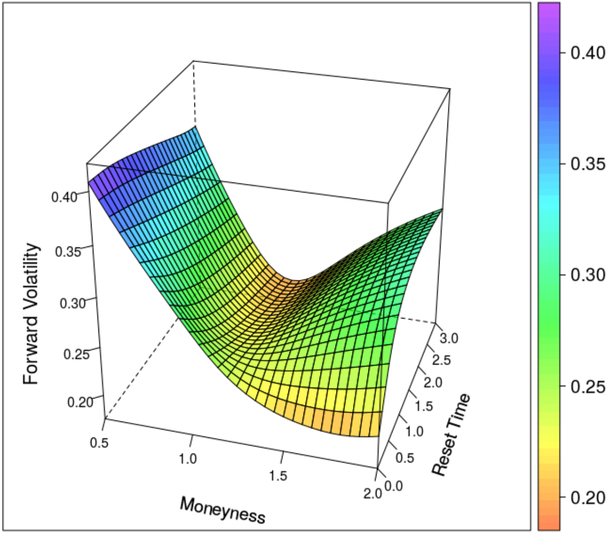
- Normal-CLV process parameters are given by

$$\kappa = 1.0, \theta = 0.1, \sigma = 0.5, x_0 = 0.1, n = 10$$

- Implied volatility of an forward starting European option with maturity date T_2 six month after the reset date T_1

$$\text{Payoff} = \max(0, S_{T_2} - \alpha S_{T_1})$$

Normal-CLV: Forward Volatility Skew



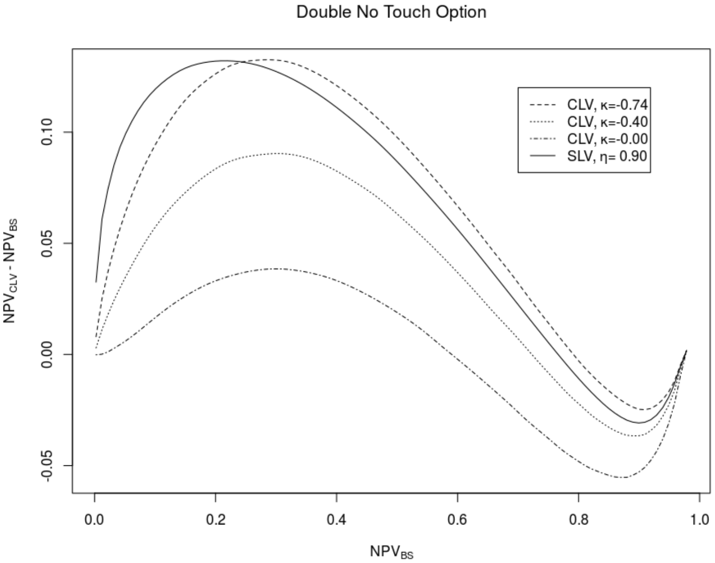
Normal-CLV: Comparison with Heston-SLV

Add leverage function $L(S_t, t)$ and mixing factor η to the Heston Model:

$$\begin{aligned}
d\ln S_t &= \left(r_t - q_t - \frac{1}{2}L(S_t,t)^2\nu_t\right)dt + L(S_t,t)\sqrt{\nu_t}\,dW_t^S \\
d\nu_t &= \kappa\left(\theta - \nu_t\right)dt + \eta\sigma\sqrt{\nu_t}\,dW_t^\nu \\
\rho dt &= dW_t^\nu dW_t^S
\end{aligned}$$

Mixing factor η tunes between stochastic and local volatility.

Normal-CLV: Comparison with Heston-SLV



Square-Root CLV Model

Choose Square-Root kernel process

$$d\nu_t = \kappa\left(\theta - \nu_t\right)dt + \sigma\sqrt{\nu_t}dW$$

The probability density function of ν_t given ν_0 is

$$\nu_t = \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\chi_d'^2\left(\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}\nu_0\right)$$

where $\chi_d'^2$ denotes the noncentral chi-square random variable with

$$d = \frac{4\theta\kappa}{\sigma^2}$$

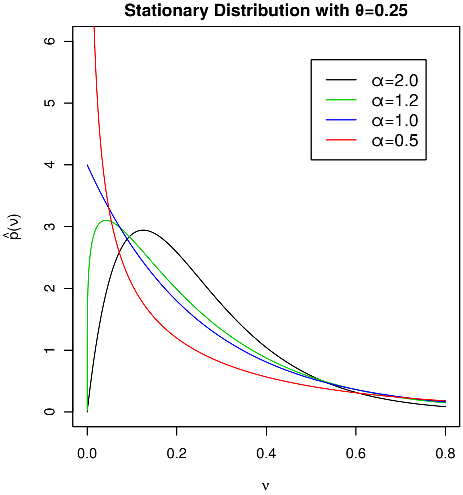
degrees of freedom and the noncentrality parameter

$$\lambda = \frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}\nu_0$$

Square-Root CLV: Stationary Kernel Distribution

$$\lim_{\nu\to 0}\hat{p}(\nu) = \begin{cases} \infty & \text{if } \alpha < 1 \\ \theta^{-1} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$

$$\text{with } \alpha = \frac{2\kappa\theta}{\sigma^2}$$



Square-Root CLV: Gaussian Quadrature

The collocation points are given by the eigenvalues of a symmetric, tridiagonal matrix with the diagonal $\{\alpha_i\}$ and the minor diagonal $\{\sqrt{\beta_i}\}$.

$$
\begin{aligned}
z_{k,i} &= z_{k-1,i+1} + \alpha_k z_{k-1,i} - \beta_k z_{k-2,i} \\
\alpha_{k+1} &= \frac{z_{k-1,k}}{k-1, k-1} - \frac{z_{k,k+1}}{z_{k,k}} \\
\beta_{k+1} &= \frac{z_{k,k}}{z_{k-1,k-1}} \\
z_{-1,i} &= 0 \\
z_{0,i} &= \mu_i = \int_0^\infty x^i \chi'^2_{d,\lambda}(x) dx \\
\alpha_1 &= -\frac{\mu_1}{\mu_0} \\
\beta_1 &= \mu_0 = 1
\end{aligned}
$$

Square-Root CLV: Gaussian Quadrature

Mathematica can calculate the first *n* moments

$$\mu_i(d, \lambda) = \int_0^\infty x^i \chi'^2_{d,\lambda}(x) dx$$

and export it as plain C code

```
m[n_] := CForm[ Expectation[X^n, X \[Distributed]
 NoncentralChiSquareDistribution[d, lambda]] // Simplify]
```

The *i*-th moment $\mu_i(d, \lambda)$ is a polynom of order *i* in *d* and λ .

The recurrence relation becomes unstable for *i* is greater than 14. Use Boost.Multiprecision instead of `QuantLib::Real` to solve the equation.

Square-Root CLV: Calibrate Forward Dynamics

Use the parameters $\{\kappa, \theta, \sigma\}$ of the kernel process to calibrate the forward skew dynamics. The mapping function $g(t, x)$ will ensure the correct pricing of European options. The forward skew dynamics is quantified by implied volatiltiy of a set of forward starting option.

Market prices are given by a Heston model with

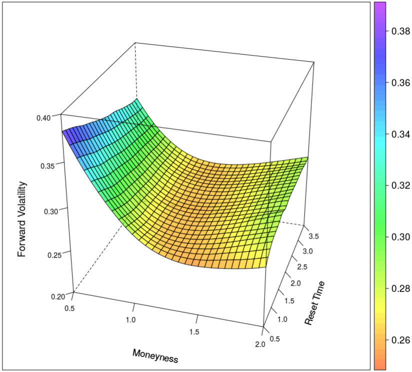
$$S_0 = 100, r = 0.1, q = 0.05,$$
$$\nu_o = 0.09, \kappa = 1.0, \theta = 0.06, \sigma = 0.4, \rho = -0.75$$

Target forward skew dynamics is defined by calibrated Heston-SLV model with mixing angle $\eta = 0.25$ and

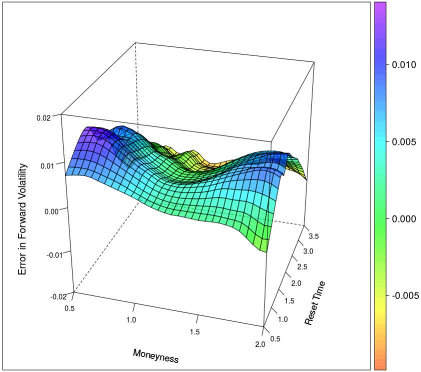
$$\nu_o = 0.09, \kappa = 1.0, \theta = 0.06, \sigma = 0.4, \rho = 0.0$$

Square-Root CLV: Calibrate Forward Dynamics

Implied volatility of an forward starting European option with maturity date six month after the reset date.



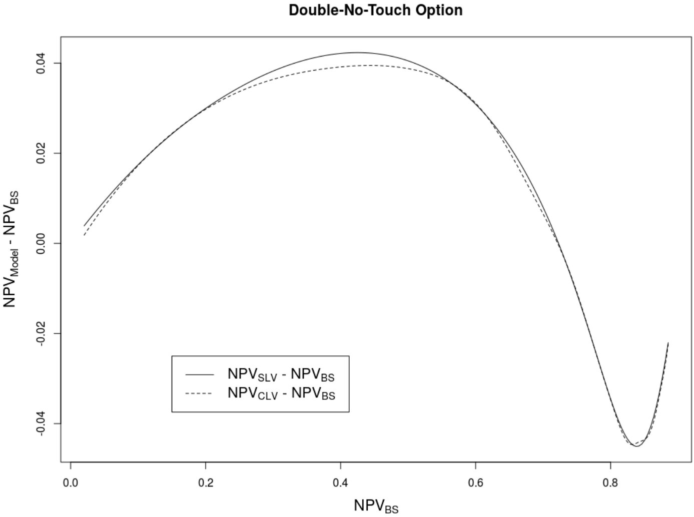
Heston-SLV Forward Volatility



Error CLV Forward Volatility

Square-Root CLV: Double-No-Touch Options

CLV forward volatility dynamics calibrated to Heston SLV dynamics.



Conclusion

- The CLV model allows for fast and accurate calibration to European option prices and efficient pricing of exotic options.
- Only the mean reversion speed influences the forward skew dynamics in the Normal-CLV model.
- The Square-Root kernel process offers more control over the forward skew dynamics.
- Dynamics can be calibrated to more complex models resulting in faster pricing

Literature

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